

# Calibrated geometries and non-perturbative superpotentials in M-theory

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**Abstract.** We consider non-perturbative effects in M-theory compactifications on a seven-manifold of  $G_2$  holonomy arising from membranes wrapped on supersymmetric three-cycles. When membranes are wrapped on associative submanifolds they induce a superpotential that can be calculated using calibrated geometry. This superpotential is also derived from compactification on a seven-manifold, to four dimensional anti-de Sitter spacetime, of eleven dimensional supergravity with non-vanishing expectation value of the four-form field strength.

## 1 Introduction

The theory of minimal surfaces has been for a long time an area of active research in mathematics. A deep novelty was the theory of calibrations, introduced by Harvey and Lawson [1] in 1982. A calibration is a closed  $k$ -form  $\psi$  on a Riemannian manifold  $M$  such that its restriction to each tangent plane of  $M$  is less than or equal to the volume of the plane. Submanifolds for which equality holds are said to be calibrated by  $\psi$ , and they have least volume in their homology class. With this defining property, minimal  $p$ -surfaces, or calibrated submanifolds, can be naturally employed to construct supersymmetric configurations of Dirichlet  $p$ -branes in string theory, as in [2–17], an approach which is also probably able to shed some light [18, 19] on the study of the moduli space of calibrated submanifolds [20, 21], whose global structure seems difficult to understand [22, 23].

Motivated by [15], where BPS solitons in the effective field theory arising from compactification of type IIA string theory, with non-vanishing Ramond–Ramond fluxes, on a Calabi–Yau fourfold, were identified by Gukov with D-branes wrapped over calibrated submanifolds in the internal Calabi–Yau space to provide a simple and geometrical derivation of the superpotential in the two dimensional field theory [24, 25], in this paper we will consider compactification of M-theory on a seven-manifold of  $G_2$  holonomy, which leads to a four dimensional field theory with  $N = 1$  supersymmetry, and apply ideas, as in [15], of the theory of calibrations to understand the superpotential.

Superpotentials induced by membrane instantons in M-theory on a seven-manifold have previously been considered from the point of view of geometric engineering of field theories by Acharya in [26], where fractional mem-

brane instantons, arising from compactification on Joyce orbifolds, are argued to generate a superpotential and, more recently, by Harvey and Moore in [14], where membranes wrapped on rigid supersymmetric three-cycles are shown to induce non-zero corrections to the superpotential, that can be expressed in terms of topological invariants of the three-cycle. In this paper we will argue how the allowed calibrations in  $G_2$  holonomy seven-manifolds imply that the only contributions to the superpotential in the four dimensional field theory will come from membranes whose worldvolume has been wrapped on supersymmetric three-cycles of the internal manifold, and exclude possible corrections from eleven dimensional five-branes, in the same way as in [15] various D-branes in type IIA string theory are shown to contribute to the superpotential in the two dimensional  $N = (2, 2)$  field theory, when wrapping calibrated submanifolds in the Calabi–Yau fourfold. However, we will not consider compactifications to four dimensional Minkowski spacetime, as in [14], but to four dimensional anti-de Sitter space, to avoid the constraint that the fourth-rank tensor field strength must be vanishing [27]. This will allow for a nice geometrical picture for the superpotential.

Relevant notions on calibrated geometries, and its realizations in supersymmetric compactifications, are described in Sect. 2. Section 3 is devoted to the construction of the superpotential induced by membrane instantons, identifying instantons in the four dimensional field theory with membranes whose worldvolume is wrapped over associative submanifolds in the  $G_2$  holonomy seven-manifold. The expression for the superpotential is justified in Sect. 4 by compactification of eleven dimensional supergravity, with non-trivial fourth-rank tensor field strength, on a manifold with  $G_2$  holonomy to a four dimensional anti-de Sitter spacetime. In Sect. 5 we present some con-

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cluding remarks and possible implications of the work presented here.

## 2 Supersymmetry and the calibration bound

In this section we will review some generalities about calibrated geometry<sup>1</sup>, and its relation to supersymmetric compactifications, and introduce some results and definitions relevant to the rest of the paper.

Given a Riemannian manifold  $M$ , with metric  $g$ , an oriented tangent  $k$ -plane  $V$  on  $M$  is a vector subspace  $V$  of the tangent space  $T_p M$  to  $M$ , with  $\dim(V) = k$ , equipped with an orientation. The restriction  $g|_V$  is the Euclidean metric on  $V$ , and allows to define, together with the orientation on  $V$ , a natural volume form  $\text{Vol}_V$  on the tangent space  $V$ .

A closed  $k$ -form  $\psi$  is said to be a *calibration on  $M$*  if for every oriented  $k$ -plane  $V$  on  $M$  it is satisfied

$$\psi|_V \leq \text{Vol}_V, \quad (2.1)$$

where by  $\psi|_V$  we mean

$$\psi|_V = \alpha \cdot \text{Vol}_V \quad (2.2)$$

for some  $\alpha \in \mathbf{R}$ , so that condition (2.1) holds only if  $\alpha \leq 1$ .

Now let  $N$  be an oriented submanifold of  $M$ , with dimension  $k$ , so that each tangent space  $T_p N$ , for  $p \in N$ , is an oriented tangent  $k$ -plane. The submanifold  $N$  is called a *calibrated submanifold* with respect to the calibration  $\psi$  if

$$\psi|_{T_p N} = \text{Vol}_{T_p N} \quad (2.3)$$

for all  $p \in N$ . It is easy to show that calibrated manifolds have minimal area in their homology class [1]; to see this, let us denote by  $[\psi]$  the de Rham cohomology class,  $[\psi] \in H^k(M; \mathbf{R})$ , and by  $[N]$  the homology class,  $[N] \in H_k(M; \mathbf{R})$ . Then,

$$[\psi] \cdot [N] = \int_{p \in N} \psi|_{T_p N}. \quad (2.4)$$

But condition (2.1) implies

$$\int_{p \in N} \psi|_{T_p N} \leq \int_{p \in N} \text{Vol}_{T_p N} = \text{Vol}(N), \quad (2.5)$$

so that  $\text{Vol}(N) \geq [\psi] \cdot [N]$ . Equality holds for calibrated submanifolds [1].

In this paper we will be interested in calibrated geometries in seven dimensional Joyce manifolds with  $G_2$  holonomy so that  $M$ , in what follows, will be a compact seven-manifold,  $M = X_7$ , with a torsion free structure, that we will denote by  $\Psi^{(3)}$  (the holonomy group of the metric associated to  $\Psi^{(3)}$  is  $G_2$  if and only if  $\pi_1(X_7)$  is finite [29]).

The covariantly constant three-form  $\Psi^{(3)}$  constitutes the *associative calibration*, with respect to the set of all associative three-planes, which are the canonically oriented imaginary part of any quaternion subalgebra of  $\mathbf{O}$ . Calibrated submanifolds with respect to  $\Psi^{(3)}$  will, in what follows, be denoted by  $S_\Psi$ , and referred to as *associative submanifolds*. The Hodge dual to the associative calibration is a four-form  $*\Psi^{(4)}$ , known as the *coassociative calibration*, responsible for *coassociative submanifolds*.

The existence of associative submanifolds, volume minimizing, can be directly understood from the point of view of compactifications of M-theory on manifolds with  $G_2$  holonomy [3]. Let us see how this comes about. The low energy limit of M-theory is expected to be eleven dimensional supergravity, which is a theory that contains membrane solutions, described by the action [30,31]<sup>2</sup>

$$S_{\text{Membrane}} = \int d^3\sigma \sqrt{h} \left[ \frac{1}{2} h^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N g_{MN} - \frac{1}{2} - i\bar{\theta} \Gamma^\alpha \nabla_\alpha \theta + \frac{i}{3!} \epsilon^{\alpha\beta\gamma} C_{MNP} \partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P + \dots \right], \quad (2.6)$$

where  $X^M(\sigma^i)$  represents the membrane configuration,  $\theta$  is an eleven dimensional Dirac spinor and  $h_{\alpha\beta}$  is the induced metric on the worldvolume of the membrane (upper case latin indices are defined in eleven dimensions, while Greek indices  $\alpha, \beta, \gamma$  label coordinates on the worldvolume).

On the membrane fields, global fermionic symmetries act as

$$\begin{aligned} \delta_\eta \theta &= \eta, \\ \delta_\eta X^M &= i\bar{\eta} \Gamma^M \theta, \end{aligned} \quad (2.7)$$

where  $\eta$  is an eleven dimensional constant anticommuting spinor. But the theory is also invariant under local fermionic transformations,

$$\begin{aligned} \delta_\kappa \theta &= 2P_+ \kappa(\sigma), \\ \delta_\kappa X^M &= 2i\bar{\theta} \Gamma^M P_+ \kappa(\sigma), \end{aligned} \quad (2.8)$$

with  $\kappa$  some eleven dimensional spinor, and  $P_+$  a projection operator [32]; the operators  $P_\pm$  are defined as

$$P_\pm = \frac{1}{2} \left( 1 \pm \frac{i}{3!} \epsilon^{\alpha\beta\gamma} \partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P \Gamma_{MNP} \right), \quad (2.9)$$

and can be easily shown to satisfy  $P_\pm^2 = P_\pm$ ,  $P_+ P_- = 0$  and  $P_+ + P_- = 1$ .

As bosonic membrane configurations break all the global supersymmetries generated by  $\eta$ , unbroken supersymmetry remains only if there is a spinor  $\kappa(\sigma)$  such that

$$\delta_\kappa \theta + \delta_\eta \theta = 2P_+ \kappa(\sigma) + \eta = 0, \quad (2.10)$$

<sup>1</sup> For a more detailed and formal treatment on calibrated geometry we refer the reader to the original reference [1], or [28] for a collection of results

<sup>2</sup> We will chose units such that the eleven dimensional Planck length is set to one

or, equivalently,

$$P_- [2P_+ \kappa(\sigma) + \eta] = P_- \eta = 0. \quad (2.11)$$

If the eleven dimensional theory is compactified on a seven-manifold of  $G_2$  holonomy, finding a covariantly constant spinor  $\eta$  satisfying condition (2.11) is equivalent to the existence of a supersymmetric three-cycle in the seven-manifold, which is precisely the associative submanifold [3]. To prove this, we will formally write the covariantly constant three-form (the associative calibration) as

$$\Psi^{(3)} = \frac{1}{3!} \psi_{mnp} dX^m \wedge dX^n \wedge dX^p, \quad (2.12)$$

and decompose the eleven dimensional spinor  $\eta$  in terms of a four dimensional spinor,  $\epsilon$ , and the covariantly constant spinor  $\xi$  on the manifold with  $G_2$  holonomy (this spinor is unique up to scale),

$$\eta = \epsilon \otimes \xi. \quad (2.13)$$

Now, if we chose a normalization such that the action of the seven dimensional gamma matrices on the covariantly constant spinor of the internal manifold is

$$\gamma_{mnp} \xi = \psi_{mnp} \xi, \quad (2.14)$$

the condition (2.11) for unbroken supersymmetry becomes

$$\frac{1}{2} \left( 1 - \frac{i}{3!} \epsilon^{\alpha\beta\gamma} \partial_\alpha X^m \partial_\beta X^n \partial_\gamma X^p \psi_{mnp} \right) \xi = 0, \quad (2.15)$$

which is equivalent to [3]

$$\frac{1}{3!} \partial_{[\alpha} X^m \partial_\beta X^n \partial_{\gamma]} X^p \psi_{mnp} = \epsilon_{\alpha\beta\gamma}, \quad (2.16)$$

so that the pull back of the three-form is proportional to the volume element<sup>3</sup>.

Condition (2.16) implies that the membrane worldvolume, wrapped on the three-cycle calibrated by  $\Psi^{(3)}$ , has been minimized, as can be easily seen from the inequality

$$\int d^3 \sigma \sqrt{h} (P_- \xi)^\dagger (P_- \xi) \geq 0, \quad (2.17)$$

as in [2]. Using the projector (2.9), and the fact that  $P_-^\dagger P_- = P_-$ , (2.17) becomes

$$\int d^3 \sigma \sqrt{h} \bar{\xi} \xi \geq \int d^3 \sigma \bar{\xi} \frac{i}{3!} \epsilon^{\alpha\beta\gamma} \partial_\alpha X^m \partial_\beta X^n \partial_\gamma X^p \psi_{mnp} \xi, \quad (2.18)$$

or  $V_3 \geq \int_{S_\Psi} \Psi^{(3)}$ . The bound is saturated if and only if  $P_- \xi = 0$ , which is precisely the condition for unbroken supersymmetry.

<sup>3</sup> An identical condition can be similarly derived for the coassociative calibration [3],

$$\frac{1}{4!} \partial_{[\alpha} X^m \partial_\beta X^n \partial_\gamma X^p \partial_{\delta]} X^q \psi_{mnpq} = \epsilon_{\alpha\beta\gamma\delta}$$

### 3 Associative calibrations and superpotential for M-theory compactifications

M-theory compactification on a seven-manifold with  $G_2$  holonomy produces a four dimensional field theory with  $N = 1$  supersymmetry. At low energy, trusting eleven dimensional supergravity as an approximation to M-theory, the effective four dimensional supergravity theory describing the massless modes is  $N = 1$  supergravity coupled to  $b_2$  vector multiplets, and  $b_3$  chiral multiplets, where  $b_2$  and  $b_3$  are Betti numbers of  $X_7$  [33]. However, in this paper we will not be interested in this, from the point of view of physics, relatively poor spectrum, or possible items of more interest when singularities are allowed in the seven-manifold, as in [26]. What we will wonder about is the generation of a non-perturbative superpotential, arising in the effective four dimensional field theory from M-theory effects, applying the ideas in [15].

In [15], the generation of a superpotential in the two dimensional theories obtained when compactifying type IIA string theory on Calabi–Yau fourfold with background Ramond–Ramond fluxes was considered, identifying BPS solitons in the field theory with D-branes wrapped over calibrated submanifolds in the internal manifold. In this section, following the same reasoning, we will wrap the M-theory membrane worldvolume over some associative submanifold in the seven-manifold  $X_7$ ,  $S_\Psi \in H_3(X_7, \mathbf{Z})$ , which is a supersymmetric three-cycle, defined through condition (2.16). This state, from the point of view of the four dimensional field theory, is an instanton. The flux of the fourth-rank antisymmetric field strength,  $F = dC$ , associated to the membrane, over the “complement”  $(S_\Psi)^\perp$  in  $X_7$ , which will be some four dimensional manifold,  $Y_4$ , jumps by one when crossing the membrane. To understand this, two points  $P$  and  $Q$  can be chosen to lie at each side of the three-cycle  $S_\Psi$ , with the membrane worldvolume wrapped on it. If  $T$  is a path joining these two points, and it is chosen to intersect the membrane at a single point, the five-manifold  $Y_4 \times T$  will also intersect the membrane at a single point. Now, as  $dF/(2\pi)$  is a delta function for the membrane,

$$\int_{Y_4 \times T} \frac{dF}{2\pi} = 1, \quad (3.1)$$

or, once Stoke’s theorem is employed,

$$\int_{Y_4 \times P} \frac{F}{2\pi} - \int_{Y_4 \times Q} \frac{F}{2\pi} = 1. \quad (3.2)$$

Hence, the field theory vacua connected by the instanton solution will correspond to different four-form fluxes,  $F_1$  and  $F_2$ ; the variation  $\Delta F/(2\pi)$  is then Poincaré dual to the homology class of the supersymmetric three-cycle,  $[S_\Psi]$ . But the amplitude for the configuration of the membrane is proportional to the volume of the associative submanifold,  $S_\Psi$ ,

$$\int_{S_\Psi} \Psi^{(3)}, \quad (3.3)$$

the constant of proportionality being simply the membrane tension. If we denote the superpotential in the field

theory by  $W$ , the amplitude of the instanton connecting two vacua will be given by the absolute value of  $\Delta W$ ; then, when (3.2) is taken into account, (3.3) becomes

$$\Delta W = \int_{S_\psi} \Psi^{(3)} = \frac{1}{2\pi} \int_{X_7=S_\psi \times Y_4} \Psi^{(3)} \wedge \Delta F^{(4)}, \quad (3.4)$$

so that we expect a superpotential

$$W = \frac{1}{2\pi} \int_{X_7} \Psi^{(3)} \wedge F^{(4)} \quad (3.5)$$

for compactification of M-theory on a seven-manifold of  $G_2$  holonomy. Expression (3.5) is the analogous of the untwisted and twisted chiral superpotentials obtained in [15] for compactification of type IIA string theory in a Calabi–Yau fourfold with non-vanishing Ramond–Ramond fluxes.

We could now also wonder about the possible contribution of M-theory five-branes to the superpotential. In fact, the result that  $\pi_1(X_7)$  is finite if and only if the holonomy is in  $G_2$  [29], implies that  $H_6(X_7; \mathbf{Z})$  is also finite, so that we might expect instanton effects coming from the five-brane. However, there is no six dimensional calibrated submanifold in  $X_7$ , because the only calibrations are  $\Psi^{(3)}$  and  $*\Psi^{(4)}$ , so that there will be no BPS five-brane instanton contribution to the superpotential.

## 4 Eleven dimensional supergravity on seven-manifolds

In this section we will present evidence for the superpotential (3.5) from supersymmetric compactification of M-theory on  $\text{AdS}_4 \times X_7$ . Compactification of eleven dimensional supergravity on a seven-manifold is very restrictive, because the four dimensional non-compact spacetime is a supersymmetric Minkowski space only if all components of the fourth-rank antisymmetric field strength vanish,  $F = 0$ , and the internal manifold is Ricci-flat [27]. However, if some of these conditions are relaxed, more general compactifications can be performed. In [34,35], compactification of eleven dimensional supergravity to four dimensional anti-de Sitter spacetime was shown to allow for a four-form field strength proportional to the cosmological constant of the external space, and which therefore vanishes for compactifications to Minkowski spacetime. The requirements on the four-form  $F$ , assuming that supersymmetry remains unbroken, with zero cosmological constant, for compactification of M-theory with a warp factor on a Calabi–Yau fourfold have been obtained in [36], and extended to a three dimensional anti-de Sitter external space in [25]. The analysis of this section, where compactification of M-theory to a four dimensional spacetime, with non-vanishing cosmological constant, on an internal manifold with  $G_2$  holonomy will be considered, will hence follow closely those in [36,25,15].

The bosonic form of the eleven dimensional effective action looks like

$$S = \frac{1}{2} \int d^{11}x \sqrt{g} \left[ R - \frac{1}{2} F^{(4)} \wedge *F^{(4)} \right.$$

$$\left. - \frac{1}{6} C^{(3)} \wedge F^{(4)} \wedge F^{(4)} - C^{(3)} \wedge I^{(8)} \right], \quad (4.1)$$

where the gravitational Chern–Simons correction, associated to the sigma-model anomaly of the six dimensional five-brane worldvolume [37], can be expressed in terms of the Riemann tensor [38],

$$I_8 = -\frac{1}{768} (\text{tr} R^2)^2 + \frac{1}{192} \text{tr} R^4. \quad (4.2)$$

The complete action is invariant under local supersymmetry transformations [39]

$$\begin{aligned} \delta_\eta e_M^A &= i\bar{\eta} \Gamma^A \psi_M, \\ \delta_\eta C_{MNP} &= 3i\bar{\eta} \Gamma_{[MN} \psi_{P]}, \\ \delta_\eta \psi_M &= \nabla_M \eta - \frac{1}{288} (\Gamma_M^{PQRS} - 8\delta_M^P \Gamma^{QRS}) F_{PQRS} \eta, \end{aligned} \quad (4.3)$$

where  $e_M^A$  is the elfbein,  $\psi_M$  the gravitino,  $\eta$  an eleven dimensional anticommuting Majorana spinor, and  $\nabla_M$  the covariant derivative, involving the Christoffel connection.

A supersymmetric configuration exists if and only if the above transformations vanish for some spinor  $\eta$ . If the background is Lorentz covariant in the four dimensional spacetime, the background spinor  $\psi_M$  must vanish, so that  $e_M^A$  and  $C_{MNP}$  are unchanged by the supersymmetry transformation. Hence, the only constraint that remains to be imposed is

$$\nabla_M \eta - \frac{1}{288} (\Gamma_M^{PQRS} - 8\delta_M^P \Gamma^{QRS}) F_{PQRS} \eta = 0. \quad (4.4)$$

The most general metric, maximally symmetric, for compactification on  $X_7$ ,

$$ds_{11}^2 = \Delta(x^m)^{-1} (ds_4^2(x^\mu) + ds_7^2(x^m)), \quad (4.5)$$

includes a scalar function called the warp factor,  $\Delta(x^m)$ , depending on the internal dimensions. We are now choosing a notation such that the eleven dimensional upper case indices split into four dimensional Greek indices, while Latin indices label the set of coordinates tangent to the internal manifold. With the choice (4.5) for the metric, condition (4.4) becomes [36],

$$\begin{aligned} \nabla_M \eta - \frac{1}{4} \Gamma_M^N \partial_N (\log \Delta) \eta \\ - \frac{\Delta^{3/2}}{288} (\Gamma_M^{PQRS} - 8\delta_M^P \Gamma^{QRS}) F_{PQRS} \eta = 0. \end{aligned} \quad (4.6)$$

Now, let us decompose the gamma matrices in the convenient  $11 = 7 + 4$  split,

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \otimes \mathbb{I}, \\ \Gamma^m &= \gamma^5 \otimes \gamma^m, \end{aligned} \quad (4.7)$$

where  $\gamma^5 = (i/4!) \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$  is the four dimensional chirality operator, so that  $(\gamma^5)^2 = +\mathbb{I}$ , and  $\gamma^m$  are chosen such that  $(i/7!) g^{1/2} \epsilon_{m_1 \dots m_7} \gamma^{m_1} \dots \gamma^{m_7} = +\mathbb{I}$ . We will also

choose, as in [35], the most general covariant form for the fourth-rank antisymmetric tensor field strength<sup>4</sup>

$$\begin{aligned} F_{\mu\nu\rho\sigma} &= m\epsilon_{\mu\nu\rho\sigma}, \\ F_{\mu\nu\rho s} &= F_{\mu\nu rs} = F_{\mu nrs} = 0, \end{aligned} \quad (4.8)$$

with

$$F_{mnpq} \text{arbitrary}, \quad (4.9)$$

where  $m$  can depend upon the extra (internal) dimensions, and we have a decomposition for the supersymmetry parameter,  $\eta = \epsilon \otimes \zeta$ .

With the split (4.7), and the above ansatz for  $F$ , the  $\mu$ -component of the supersymmetry condition (4.6) becomes

$$\begin{aligned} \nabla_\mu \eta - \frac{1}{4} \gamma_\mu \gamma^5 \otimes \gamma^n \partial_n (\log \Delta) \eta - \frac{\Delta^{3/2}}{288} \gamma_\mu \gamma^{mnpq} F_{mnpq} \eta \\ + \frac{1}{6} \Delta^{3/2} i m \gamma_5 \gamma_\mu \eta = 0. \end{aligned} \quad (4.10)$$

When  $\epsilon$  is a four dimensional anti-commuting Killing spinor satisfying  $\nabla_\mu \epsilon = (\Lambda/2) \gamma_\mu \epsilon$ , (4.10), with the decomposition  $\eta = \epsilon \otimes \zeta$ , leads to the solution

$$\begin{aligned} 144 \Lambda \zeta &= \Delta^{3/2} F_{mnpq} \gamma^{mnpq} \zeta, \\ m \zeta &= i \gamma^n \partial_n \Delta^{-3/2} \zeta. \end{aligned} \quad (4.11)$$

Similarly, with the decomposition (4.7), the  $m$ -component of condition (4.6) becomes, after some gamma matrices algebra<sup>5</sup>,

$$\begin{aligned} \nabla_m \eta - \frac{\Lambda}{2} \gamma_m \eta + \frac{1}{4} \partial_m (\log \Delta) \eta - \frac{3}{8} \partial_n (\log \Delta) \gamma_m^n \eta \\ + \frac{\Delta^{3/2}}{24} F_{mpqr} \gamma_5 \gamma^{pqr} \eta = 0, \end{aligned} \quad (4.12)$$

where we have made use of (4.11). As in [36, 25], the transformed quantities

$$\begin{aligned} \tilde{g}_{mn} &= \Delta^{-3/2} g_{mn}, \\ \rho &= \Delta^{1/4} \eta, \end{aligned} \quad (4.13)$$

lead (4.12) into the simpler form

$$\tilde{\nabla}_m \rho - \frac{\Lambda}{2} \Delta^{3/4} \tilde{\gamma}_m \rho + \frac{1}{24} \Delta^{-15/4} F_{mpqr} \tilde{\gamma}_5 \tilde{\gamma}^{pqr} \rho = 0. \quad (4.14)$$

<sup>4</sup> In compactifications of string theory or M-theory on a Calabi–Yau fourfold a global anomaly, given by the Euler number of the Calabi–Yau fourfold, arises [40, 41]. This anomaly can be cancelled if non-zero background fluxes are allowed, and/or strings or membranes are introduced filling, respectively, two or three dimensional spacetime, for compactification of string theory or M-theory. These strings, or membranes, are represented by maximally symmetric tensor fields. However, for seven-manifolds there is no such interpretation for the ansatz of the first equation in (4.8)

<sup>5</sup> A complete collection of commutators and anticommutators, and useful gamma matrices identities, can be found in the appendices in [27] and [42], where the  $11 = 7 + 4$  split is also considered

Now, if we decompose the eleven dimensional spinor  $\rho$  as  $\xi \otimes \varepsilon$ , with  $\xi$  chosen to be the covariantly constant spinor on the seven-manifold of  $G_2$  holonomy, and we use the fact that the associative calibration  $\Psi^{(3)}$  satisfies relation (2.14),  $\gamma^{pqr} \xi = \psi^{pqr} \xi$ , then the components of (4.14) reproduce, when integrated over the volume of  $X_7$ , the form of the superpotential proposed in (3.5), with a coefficient related to the warp factor, depending hence on the internal dimensions, as a consequence of the ansatz (4.8), and with the cosmological constant  $\Lambda$  identified with the vacuum value of the superpotential,  $W$ , in the same way as the mass, and the twisted mass, are identified in [15] with the vacuum values of  $W$  and  $\tilde{W}$ . Note, however, the appearance of the chirality operator in expression (4.14).

## 5 Conclusions

In this paper we have investigated the generation of a non-perturbative superpotential in the four dimensional field theories obtained from compactification of M-theory on a seven-manifold with  $G_2$  holonomy. The allowed calibrations in these manifolds imply that the only BPS instantons are those obtained wrapping the worldvolume of the M-theory membrane over associative submanifolds, which are the analogous of special Lagrangian cycles in Calabi–Yau manifolds, while instantons coming from five-branes wrapping six-cycles in the seven-manifold are not BPS states. An analysis relying on calibrated submanifolds can also be used to study compactification of M-theory on a Calabi–Yau fourfold, where five-branes wrapped over (complex) codimension one divisors  $D$  are, however, known to contribute to the superpotential in the three dimensional field theory arising upon compactification of M-theory on the fourfold if the divisor satisfies the topological requirement that  $\chi(\mathcal{O}_D, D) = 1$  [43]. This precise condition can probably be related to the fact that fourfolds with  $SU(4)$  holonomy admit, besides Cayley calibrations, what are known as Kähler calibrations, which are obtained when considering powers of the complexified Kähler form,

$$\Psi = \frac{1}{p!} \mathcal{K}^p, \quad (5.1)$$

because the fact that  $\mathcal{K}$  is covariantly constant ensures that  $\Psi$ , as defined in (4.13), is also covariantly constant. Submanifolds calibrated by  $\Psi$  are complex submanifolds, of dimension  $p$ , so that five-branes will become BPS instantons once they are wrapped over some (complex) three dimensional manifold, calibrated by  $(1/3!) \mathcal{K}^3$ .

In [23], singularities of special Lagrangian three-cycles and their compactness properties were studied in detail, and a topological invariant, counting special Lagrangian homology three-spheres, was proposed. If an equivalent analysis can be repeated on seven-manifolds with  $G_2$  holonomy, then an analogous topological invariant can probably be constructed, counting associative submanifolds or, which is more interesting physically, counting the number of membranes, or instantons, in the homology class  $[S_\Psi]$ , an idea closely related to the Gromov–Witten

invariants, counting pseudo-holomorphic curves in symplectic manifolds. We hope to address this study in a subsequent paper.

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